

The Optimal Retailer's Ordering Policies with Inventory-Dependent Demand and Limited Storage Capacity under Trade Credit Financing

Yuanguang Zhong*, Yongwu Zhou, Qionglin Liao, Jinsen Guo
 School of Business Administration
 South China University of Technology, Guangzhou, China
 *EMAIL: hitscut@163.com

Abstract: Traditionally, inventory models with trade credit policy deal with contain demand or merely dependent on the retailing price. Therefore, this paper tries to incorporate the retailer's inventory-dependent demand and storage space limited in the retailer's inventory model, which will make the decision-maker of inventory system to know whether to rent RW and how to order. Two easy-to-use theorems are developed to efficiently determine the optimal inventory policy for the retailer. Finally, we deduce Goyal's model [3] as a special case.

Keywords: Two-warehouse system, inventory-dependent demand, trade credit.

I. Introduction

In the based EOQ model, it is assumed that the retailer must pay for the items as soon as he received them from a supplier. However, many suppliers usually allow in practice their retailers a trade credit for settling the account without any interest charged. For example, Wal-Mart, the largest retailer in the world, has used trade credit as a larger source of capital than bank borrowings. Also, Aaronson et al. (2004)[1] reported that "60.8 percent of firms had outstanding credit from suppliers". This type of trade credit is equivalent to offering the retailers short-term interest-free finance in stock. Hence, the trade credit should affect the retailer's conduct of order significantly. In this regard, a lot of research papers appeared which deal with the inventory problems with trade credit intended to link marketing, financing as well as operations. For example, Haley and Higgins (1973)[2], Goyal (1985)[3] studied the effect of credit period on the optimal inventory policy, from the retailer's point of view. Chung (1998)[4] simplified the search for an optimal solution to Goyal's model (1985). Teng (2002)[5] then amended Goyal model (1985) to consider the difference between unit price and unit cost. Zhou (1997) [6]discussed the impact of different rules for delay in payment on the retailer's order policy. Recently, Chung and Liao (2009)[7] developed a new inventory model where the conditions of using a DCF approach and trade credit are dependent on the quantity ordered.

The previous papers assumed that the demand was a known constant. Hence, they ignored the effects of the credit period on the demand volume. In many actual situations, however, the supplier's intention of offering the trade credit period to

retailers is to stimulate the demand for products. In order to reflect it in inventory models with trade credit period permitted, Teng et al. (2005)[8], Sheen and Tsao (2007)[9] have employed price-sensitive demand. Chang et al. (2001)[10] proposed a replenishment model for deteriorating items with permissible delay in payments and linear trend demand during a finite time horizon. They further extended this model in another paper (2002)[11] to consider monetary time-value. In addition, it had been noted, especially in the retailer industry, that holding higher inventory level will probably make the retailer sell more items. Under this situation, the demand rate should depend on the inventory level. For instance, Liao et al. (2000)[12] developed an inventory model for deteriorating items with initial-stock-dependent consumption rate when a delay in payment is permissible. Recently, Min et al. (2010)[13] develops a lot-sizing model for deteriorating items with a current-stock-dependent demand and delay in payments.

As we known, in practice, trade credit policy encourages the retailer to order large quantities because a delay of payments indirectly reduces inventory cost. In addition, for the system with inventory-level-dependent demand rate, holding large piles of goods will lead the customers to buy more. This will also make the system replenish more goods than can be stored in own warehouse. However, the models listed above assume that the available warehouse has unlimited capacity in those models. Hence, inventory models should be extended to the situation with multiple warehouses. Hartely (1976)[14] first proposed this kind of system. In general, the holding cost in RW is higher than that in OW. Hence, items in RW are first transferred to OW to meet the demand until the stock level in RW drops to zero and then items in OW are released. Several researchers have extended this filed such as Bhunia and Maiti (1998)[15], Zhou (1998)[16], Kar et al. (2001)[17], Zhou and Yang (2005) [18] and so on. Recently, Huang (2006) [19] investigate the retailer's inventory policy under two levels of trade credit and limited storage space. Chung and Huang (2007)[20] proposed a two-warehouse inventory model for deteriorating items under permissible delay in payments. However, few inventory models with two warehouses have been found in the literature that addresses an inventory-level-dependent demand and trade credit policy.

In this paper, we also try to discuss how the retailer to order when he/she faces an inventory-dependent demand and limited storage capacity.

II. Assumption and notations

2.1 Assumption

- (1) Time horizon is infinite.
- (2) Shortage is not allowed.
- (3) The item is not damaged either physically or technically.
- (4) Time horizon is infinite, and replenishment rate is instantaneous.
- (5) The supplier's credit period begins at the time when the retailer receives the ordered items.
- (6) During the time the account is not settled, the retailer's generated sales revenue is deposited in an interest-bearing account. At the end of this period, the account is settled and the retailer starts paying for the interest charges on investment in inventory.
- (7) The OW has limited capacity of W units and the RW has unlimited capacity.
- (8) For economic reasons, the items of RW are consumed first and next the goods of OW.

2.2 Notations

- (1) p The unit selling price
- (2) A_r Order cost one order
- (3) h_r OW inventory holding cost per unit per year for the retailer, and inventory holding cost per unit per year for the supplier, excluding the cost of capital.
- (4) k RW inventory holding cost per unit per year for the retailer.
- (5) s_r The opportunity cost per unit per year at level i , excluding the holding cost, which may be measured in practice by $I_r c_r$, where c_r is the procurement unit cost and I_r the interest charges per \$ investment in inventory per year for the retailer.
- (6) g_r The opportunity gain per unit per year for the retailer, which may be estimated similarly by $I_e c_r$, where I_e is the interest earned per \$ per year for the retailer.
- (7) H_r Inventory cost per unit item per year for the retailer, $H_r = (h_r + s_r)$.
- (8) W The retailer's OW storage capacity.
- (9) T_w The rented warehouse time in years.
- (10) $I(t)$ The retailer's inventory level at time t .
- (11) I_0 The initial stock level.
- (12) $q(t)$ The market demand rate the retailer faces, which is dependent on the current inventory level and is assumed to be in the following polynomial form: $q(t) = \alpha I_0^\beta$, $0 \leq t \leq T$, where $\alpha > 0$ and $0 < \beta < 1$, are scale and shape parameters, respectively, β reflects the elasticity of the demand rate with respect to the initial-stock-level elasticity. The values of α and β are known to the supplier and the retailer.
- (14) Q The retailer's order quantity

III. Mathematic model

Based on the assumptions, and nations made in Section II, when the order quantity $Q \leq W$, the initial stock level $I_0 = Q$

and the inventory level $I(t)$ with respect to time t can be described by the following differential equation:

$$\begin{cases} \frac{dI(t)}{dt} = -\alpha Q^\beta, 0 \leq t \leq T \\ I(0) = Q, I(T) = 0 \end{cases} \quad (1)$$

Hence, the solution to Eq. (1) is

$$I(t) = \alpha Q^\beta (T - t) \quad (2) \quad \text{and} \quad T = Q^{1-\beta} / \alpha \quad (3)$$

Let $T_a = W^{1-\beta} / [\alpha(1-\beta)]$, Equation (3) yields $T > T_a$, if and only if $Q > W$;

On the other hand, when the order quantity $Q > W$, the inventory level at RW reduces due to demand for a time T_w until reaching zero. Depending on the assumption, the initial stock level is W at OW.

As described above, the variation of $I_R(t)$ with respect to time is described by the following differential equation:

$$\begin{cases} \frac{dI_R(t)}{dt} = -\alpha W^\beta, 0 < t < T_w \\ I_R(0) = Q - W, I_R(T_w) = 0 \end{cases} \quad (4)$$

The solution of Eq. (4) is

$$I_R(t) = \alpha W^\beta (T_w - t) \quad 0 \leq t \leq T_w \quad (5)$$

Based on $I_R(T_w) = 0$, we easily get $T_w = (Q - W) / \alpha W^\beta$ (6)

Let $I_{o1}(t)$ and $I_{o2}(t)$ denotes the level of inventory at OW during the time interval $(0, T_w)$ and (T_w, T) , respectively. Therefore, $I_{o1}(t) = W$, $0 \leq t \leq T_w$ and the variation of $I_{o2}(t)$ with respect to time is described by the following differential equation:

$$\begin{cases} \frac{dI_{o2}(t)}{dt} = -\alpha W^\beta, T_w < t < T \\ I_{o2}(T_w) = W, I_{o2}(T) = 0 \end{cases} \quad (7)$$

The solution of Eq. (8) is

$$I_{o2}(t) = \alpha W^\beta (T - t) \quad T_w \leq t \leq T \quad (8)$$

Based on Eq. (7) and $I_{o2}(T) = 0$, we can have $T = Q / \alpha W^\beta$ (9)
Accordingly, There are two cases to occur: (A) $T_a < M < T_a + M$;
(B) $M \leq T_a \leq T_a + M$.

(A) Suppose that $T_a < M < T_a + M$

The average profit consists of the following elements.

- (1) Ordering cost per cycle = A_r ;
- (2) Sales revenue per cycle

Case 1: $T \leq T_a$, $pQ = p(\alpha T)^{\frac{1}{1-\beta}}$

Case 2: $T > T_a$, $pQ = p\alpha W^\beta T$

- (3) Purchase cost per cycle

Case 1: $T \leq T_a$, $c_r Q = c_r (\alpha T)^{\frac{1}{1-\beta}}$

Case 2: $T > T_a$, $c_r Q = c_r \alpha W^\beta T$

(4) Inventory holding cost in RW for two cases is obtained as followings:

Case 1: $T \leq T_a$

In this case, it is not need to rent warehouse. Therefore, no stock holding cost for items in RW.

Case 2: $T > T_a$

Inventory holding cost per circle in RW

$$= k \int_0^{T_w} I_R(t) dt = \frac{k\alpha W^\beta T_w^2}{2}$$

(5) Inventory holding cost in OW for two cases is obtained as follows:

Case 1: $T \leq T_a$

Inventory holding cost per circle in OW

$$= h_r \int_0^T I(t) dt = \frac{h_r Q^{2-\beta}}{2\alpha} = \frac{1}{2} h_r \alpha^{\frac{1}{1-\beta}} T^{\frac{2-\beta}{1-\beta}}$$

Case 2: $T > T_a$

Inventory holding cost per circle in OW

$$= h_r \left[\int_0^{T_w} I_{o1}(t) dt + \int_{T_w}^T I_{o2}(t) dt \right] = h_r \left(WT_w + \frac{W^{2-\beta}}{2\alpha} \right)$$

(6) The opportunity cost per unit per circle is obtained as follows:

Case 1: $T \leq M$

In this case, no interest charges are paid for the items.

Case 2: $M < T < T_a + M$, Interest payable per circle

$$= \int_M^T I_{o2}(t) dt = \frac{1}{2} s_r \alpha W^\beta (T - M)^2$$

Case 3: $T_a + M \leq T$, Interest payable per circle

$$= s_r \left[\int_M^{T_w} I_R(t) dt + \int_M^{T_w} I_{o1}(t) dt + \int_{T_w}^T I_{o2}(t) dt \right]$$

$$= s_r \left[\frac{\alpha W^\beta (T_w - M)^2}{2} + W(T_w - M) + \frac{W^{2-\beta}}{2\alpha} \right]$$

(7) The interest earned per unit per circle is obtained as follows:

Case 1: $T \leq T_a < M$, the interest earned per circle

$$= g_r \int_0^T [Q - I(t)] dt + g_r Q(M - T)$$

$$= g_r M (\alpha T)^{\frac{1}{1-\beta}} - \frac{1}{2} g_r \alpha^{\frac{1}{1-\beta}} T^{\frac{2-\beta}{1-\beta}}$$

Case 2: $T_a < T < M$, the interest earned per circle

$$= \frac{1}{2} g_r QT + g_r Q(M - T) = g_r \alpha W^\beta T \left(M - \frac{T}{2} \right)$$

Case 3: $M \leq T$, the interest earned per circle

$$= g_r \left(\frac{\alpha W^\beta M^2}{2} \right)$$

Therefore, the annual profit for retailer can be expressed as $\Pi_r(T) = \{\text{Sales revenue} - \text{Purchase cost} - \text{Order cost} - \text{Stock-holding cost in RW} - \text{Stock-holding cost in OW} - \text{The opportunity cost}\} / T$

From the above arguments, we consider the following results:

$$\Pi_r(T) = \begin{cases} \Pi_{r1}(T), & 0 < T \leq T_a \\ \Pi_{r2}(T), & T_a < T < M \\ \Pi_{r3}(T), & M \leq T < T_a + M \\ \Pi_{r4}(T), & T_a + M \leq T \end{cases} \quad (10a-d)$$

Where

$$\Pi_{r1}(T) = \frac{1}{T} \left[(p - c_r + g_r M) (\alpha T)^{\frac{1}{1-\beta}} - A_r - \frac{1}{2} (h_r + g_r) \alpha^{\frac{1}{1-\beta}} T^{\frac{2-\beta}{1-\beta}} \right] \quad (11)$$

$$\Pi_{r2}(T) = \frac{1}{T} \left[(p - c_r + g_r M) \alpha W^\beta T - A_r - \frac{k\alpha W^\beta T_w^2}{2} - h_r \left(WT_w + \frac{W^{2-\beta}}{2\alpha} \right) - g_r \alpha W^\beta \frac{T^2}{2} \right] \quad (12)$$

$$\Pi_{r3}(T) = \left[\frac{1}{T} (p - c_r) \alpha W^\beta T - A_r - \frac{k\alpha W^\beta T_w^2}{2} - h_r \left(WT_w + \frac{W^{2-\beta}}{2\alpha} \right) - \frac{1}{2} s_r \alpha W^\beta (T - M)^2 + g_r \frac{\alpha W^\beta M^2}{2} \right] \quad (13)$$

and

$$\Pi_{r4}(T) = \frac{1}{T} \left[(p - c_r) \alpha W^\beta T - A_r - \frac{K\alpha W^\beta T_w^2}{2} - H_r \left(WT_w + \frac{W^{2-\beta}}{2\alpha} \right) + \alpha W^\beta s_r M \left(T - \frac{M}{2} \right) + g_r \frac{\alpha W^\beta M^2}{2} \right] \quad (14)$$

(B) Suppose that $M \leq T_a \leq T_a + M$.

$$\Pi_r(T) = \begin{cases} \Pi_{r1}(T), & 0 < T \leq M \\ \Pi_{r5}(T), & M < T \leq T_a \\ \Pi_{r3}(T), & T_a < T < T_a + M \\ \Pi_{r4}(T), & T_a + M \leq T \end{cases} \quad (15a-d)$$

thus the annual profit of $\Pi_{r5}(T)$ comprises the following elements:

(1) Ordering cost per cycle = A_r ;

(2) Sales revenue per cycle = $p(\alpha T)^{1/1-\beta}$

(3) Purchase cost per cycle = $c(\alpha T)^{1/1-\beta} c_r Q = c_r (\alpha T)^{\frac{1}{1-\beta}}$

(4) Inventory holding cost in RW : no stock holding cost for items in RW.

(5) Inventory holding cost in

$$OW = h_r \int_0^T I(t) dt = \frac{h_r Q^{2-\beta}}{2\alpha} = \frac{1}{2} h_r \alpha^{\frac{1}{1-\beta}} T^{\frac{2-\beta}{1-\beta}}$$

(6) The opportunity cost per unit per circle is obtained as follows:

$$= \int_M^T I(t) dt = \frac{1}{2} s_r \alpha^{\frac{1}{1-\beta}} T^{\frac{\beta}{1-\beta}} (T - M)^2$$

(7) The interest earned per unit per circle is obtained as follows:

$$= g_r \int_0^M [Q - I(t)] dt = g_r M (\alpha T)^{\frac{1}{1-\beta}} - g_r \alpha^{\frac{1}{1-\beta}} T^{\frac{\beta}{1-\beta}} M \left(T - \frac{M}{2} \right)$$

Therefore, the annual profit of $\Pi_{r5}(T)$ is obtained as follows:

$$\Pi_{r5}(T) = \frac{1}{T} \left[(p - c_r + g_r M) (\alpha T)^{\frac{1}{1-\beta}} - A_r - \frac{1}{2} H_r \alpha^{\frac{1}{1-\beta}} T^{\frac{2-\beta}{1-\beta}} + (s_r - g_r) \alpha^{\frac{1}{1-\beta}} T^{\frac{\beta}{1-\beta}} M \left(T - \frac{M}{2} \right) \right] \quad (16)$$

IV. Optimization

Before proving Theorems, we need the following lemma:

Lemma 1:

- (1) $\Pi_{r1}(T)=0$ has a unique solution T_1^* on $(0,\infty)$;
- (2) $\Pi_{r2}(T)=0$ has a unique solution T_2^* on $(0,\infty)$;
- (3) $\Pi_{r3}(T)=0$ has a unique solution T_3^* on $(0,\infty)$;
- (4) $\Pi_{r4}(T)=0$ has a unique solution T_4^* on $(0,\infty)$;
- (5) $\Pi_{r5}(T)=0$ has a unique solution T_5^* on $(0,\infty)$;

Proof. See Appendix A for detail

(A) Decision rule of the optimal replenishment time when $T_a < M < T_a + M$

In follows from Eqs (10a-b) that

$$\Pi_r(T) = \begin{cases} \Pi_{r1}(T), & 0 < T \leq T_a \\ \Pi_{r2}(T), & T_a < T < M \\ \Pi_{r3}(T), & M \leq T < T_a + M \\ \Pi_{r4}(T), & T_a + M \leq T \end{cases}$$

Obviously, $\Pi_{r1}(T_a) = \Pi_{r2}(T_a)$, $\Pi_{r2}(M) = \Pi_{r3}(M)$ and $\Pi_{r3}(T_a + M) = \Pi_{r4}(T_a + M)$. Hence $\Pi_r(T)$ is continuous on $(0,\infty)$.

Eqs. (A1-A4) yield that

$$\Pi_{r1}'(T_a) = \Delta_1/T_a^2, \quad \Pi_{r2}'(T_a) = \Delta_2/T_a^2$$

$$\Pi_{r2}'(M) = \Pi_{r3}'(M) = \Delta_3/M^2$$

and

$$\Pi_{r3}'(T_a + M) = \Pi_{r4}'(T_a + M) = \Delta_4/(T_a + M)^2$$

Let

$$\Delta_1 = \frac{\beta}{1-\beta}(p-c_r + g_r M)W + A_r - \frac{(h_r + g_r)WT_a}{2(1-\beta)}$$

$$\Delta_2 = \frac{1}{T_a^2} \left[A_r - \frac{(h_r + g_r)WT_a}{2} \right]$$

$$\Delta_3 = A_r - \frac{1}{2}(k + g_r)\alpha W^\beta M^2 + \frac{(k - h_r)WT_a}{2}$$

and

$$\Delta_4 = A_r - \frac{K\alpha W^\beta (T_a + M)^2}{2} + \frac{(k - h_r)WT_a}{2} + \frac{1}{2}(s_r - g_r)\alpha W^\beta M^2$$

Given the any value of M , we can get

$$\Delta_1 - \Delta_2 = \frac{\beta}{2(1-\beta)} [2(p-c_r + g_r M)W - (h_r + g_r)WT_a] > 0,$$

base on the **Lemma 1**, we can easily obtain $\Delta_2 > \Delta_3 > \Delta_4$.

Then, we have the following **Theorem 1**.

Theorem 1:

- (a) if $\Delta_1 < 0$, the optimal replenishment time is T_1^* ;
- (b) if $\Delta_1 \geq 0$, $\Delta_2 < 0$, the optimal replenishment time is T_a ;
- (c) if $\Delta_2 \geq 0$, $\Delta_3 < 0$, the optimal replenishment time is T_2^* ;
- (d) if $\Delta_3 \geq 0$, $\Delta_4 < 0$, the optimal replenishment time is T_3^* ;
- (e) if $\Delta_4 \geq 0$, the optimal replenishment time is T_4^* ;

Proof. See Appendix B for detail.

From theorem1, we have the retailer's optimal order cycle is $T_1^* < T_a$ as $\Delta_1 < 0$, which implies the retailer's order quantity is not more than his/her storage capacity and he/she needn't rent warehouse. In additionally, if the retailer's optimal replenishment is T_a^* as $\Delta_1 \geq 0$, and $\Delta_2 < 0$, which means his/her optimal order quantity is W and is equal to his/her storage capacity. However, if the retailer's optimal replenishment is T_2^* as $\Delta_2 \geq 0$, $\Delta_3 < 0$, T_3^* as $\Delta_3 \geq 0$, $\Delta_4 < 0$, and T_4^* as $\Delta_4 \geq 0$, then the retailer's optimal order quantity is more than his storage capacity and he/she would rent the other warehouse .

(B) Decision rule of the optimal replenishment time when $M \leq T_a \leq T_a + M$.

$$\Pi_r(T) = \begin{cases} \Pi_{r1}(T), & 0 < T \leq M \\ \Pi_{r5}(T), & M < T \leq T_a \\ \Pi_{r3}(T), & T_a < T < T_a + M \\ \Pi_{r4}(T), & T_a + M \leq T \end{cases}$$

Likewise, $\Pi_{r1}(M) = \Pi_{r5}(M)$, $\Pi_{r5}(T_a) = \Pi_{r3}(T_a)$ and $\Pi_{r3}(T_a + M) = \Pi_{r4}(T_a + M)$. Hence $\Pi_r(T)$ is continuous on $(0,\infty)$.

Eqs. (A1, A5, A3 and A4) yield that

$$\Pi_{r1}'(M) = \Pi_{r5}'(M) = \Delta_5/M^2, \quad \Pi_{r5}'(T_a) = \Delta_6/T_a^2$$

$$\Pi_{r3}'(T_a) = \Delta_7/T_a^2$$

and

$$\Pi_{r3}'(T_a + M) = \Pi_{r4}'(T_a + M) = \Delta_4/(T_a + M)^2$$

Let

$$\Delta_5 = \frac{1}{M^2} \left[\frac{\beta}{1-\beta}(p-c_r + g_r M)(\alpha M)^{\frac{1}{1-\beta}} + A_r - \frac{1}{2(1-\beta)}(h_r + g_r)\alpha^{1-\beta} M^{\frac{2-\beta}{1-\beta}} \right]$$

$$\Delta_6 = \frac{1}{T_a^2} \left[\frac{\beta}{1-\beta}(p-c_r + g_r M)W + A_r - \frac{1}{2(1-\beta)} H_r W T_a + \frac{(s_r - g_r)WM}{(1-\beta)T_a} \left(\beta(T_a - M) + \frac{M}{2} \right) \right]$$

and

$$\Delta_7 = \frac{1}{T_a^2} \left[A_r - \frac{H_r W T_a}{2} + \frac{1}{2}(s_r - g_r)\alpha W^\beta M^2 \right]$$

Given the any value of M , we can get

$$\Delta_6 - \Delta_7 = \frac{\beta}{2(1-\beta)} \left[2(p-c_r + g_r M)W - H_r W T_a + (s_r - g_r) \left(2 - \frac{M}{T_a} \right) MW \right] > 0$$

base on the **Lemma 1**, we can easily obtain

$$\Delta_5 > \Delta_6 > \Delta_7 > \Delta_4$$

Theorem 2:

- (1) if $\Delta_5 < 0$, then the optimal replenishment time is T_1^* ;
- (b) if $\Delta_5 \geq 0$, $\Delta_6 < 0$, the optimal replenishment time is T_5^* ;
- (c) if $\Delta_6 \geq 0$, $\Delta_7 < 0$, the optimal replenishment time is T_a ;
- (d) if $\Delta_7 \geq 0$, $\Delta_4 < 0$, the optimal replenishment time is T_3^* ;

(e) if $\Delta_4 \geq 0$, then the optimal replenishment time is T_4^* ;

Proof. See Appendix C for detail.

In a similar analysis in Theorem 1, we can have the following results: (i) the retailer's optimal order cycle is $T_1^* < T_a$ as $\Delta_1 < 0$ and $T_5^* < T_a$ as $\Delta_5 \geq 0$ and $\Delta_6 < 0$, which implies the retailer's order quantity is not more than his/her storage capacity and he/she needn't rent warehouse; (ii) if the retailer's optimal replenishment is T_a^* as $\Delta_6 \geq 0$ and $\Delta_7 < 0$, which means his/her optimal order quantity is W and is equal to his/her storage capacity; (iii) However, if the retailer's optimal replenishment is T_3^* as $\Delta_7 \geq 0$, $\Delta_4 < 0$, T_4^* as $\Delta_4 \geq 0$, then the retailer's optimal order quantity is more than his storage capacity and he/she would rent the other warehouse.

V. Special case

In this section, we discuss a special case (i.e. Goyal's model) and make description of this case.

When $p=c_r$, $\beta \rightarrow 0$, and $W \rightarrow \infty$ will imply that $T_a \rightarrow \infty$, by omitting the sales revenue and purchasing cost, using L'Hopital's rule, Eqs (11) and (16) yield that

$$\lim_{\beta \rightarrow 0^+} \lim_{W \rightarrow \infty} \Pi_{r1}(T) = \frac{1}{T} \left[\frac{g_r \alpha}{2} (2MT - T^2) - \frac{h_r \alpha T^2}{2} - A_r \right] \equiv TC_1(T)$$

and

$$\lim_{\beta \rightarrow 0^+} \lim_{W \rightarrow \infty} \Pi_{r5}(T) = \frac{1}{T} \left[\frac{g_r \alpha M^2}{2} - \frac{s_r \alpha}{2} (T - M)^2 - \frac{h_r \alpha T^2}{2} - A_r \right] \equiv TC_2(T)$$

Then (15a-d) will be reduced as follows:

$$TC(T) = \begin{cases} TC_1(T) & 0 \leq T \leq M \\ TC_2(T) & T \geq M \end{cases}$$

Then the above equations are consistent with Eqs. (4) and (1) in Goyal's model [3], respectively. Hence, Goyal's model will be a special case of this paper.

VI. Conclusions

In this paper, we develop an EOQ model under permissible delay in payment. The primary differences of this paper as compared to previous studies is that we introduce a generalized inventory model by relaxing the traditional EOQ model under trade credit financing in the following ways: (1) the demand of items is dependent on the retailer's initial stock level, (2) the retailer storage space is limited, and (3) the maximizing profit is used as the objective to find the optimal replenishment policy. Goyal's model, which is the first extended model under trade credit policy, is a special case in this paper. In addition, we established the necessary and sufficient conditions for the unique optimal replenishment interval and constructed two theoretical results, which is easy-to-use for the retailer. The presented model can be extended to some more practical situations,

such as probabilistic demand, allowable shortages, or finite replenishment rate etc.

Acknowledgement

This study is supported by funds of NSFC (Grant No. 70771034; 70971041) and The Important Project Guangdong Province Major Research Base for Humanities and Social Sciences (Grant No. 08JDXM63003).

Appendices

Appendix A. Proof of Lemma 1:

(a) To proof the first part in Lemma 1, taking the derive of $\Pi_{r1}(T)$ in (11) with respect to T will be give $\Pi'_{r1}(T)=f_1(T)/T^2$, Where

$$f_1(T) = \left[\frac{\beta}{1-\beta} (p-c_r + g_r M) (\alpha T)^{\frac{1}{1-\beta}} + A_r - \frac{1}{2(1-\beta)} (h_r + g_r) \alpha^{\frac{1}{1-\beta}} T^{\frac{2-\beta}{1-\beta}} \right] \quad (A1)$$

Further, taking the second derivative of $\Pi_{r1}(T)$ in (10) with respect to T gives

$$\Pi''_{r1}(T) = -\frac{1}{T^3} \left[\frac{\beta(1-2\beta)}{(1-\beta)^2} (p-c_r + g_r M) (\alpha T)^{\frac{1}{1-\beta}} + 2A_r + \frac{\beta}{2(1-\beta)^2} (h_r + g_r) \alpha^{\frac{1}{1-\beta}} T^{\frac{2-\beta}{1-\beta}} \right] < 0$$

Hence, $\Pi'_{r1}(T)=0$ has a unique solution T_1^* on $(0, \infty)$.

(b) Similarly, from (12), we have $\Pi'_{r2}(T)=f_2(T)/T^2$

$$f_2(T) = A_r - \frac{k\alpha W^\beta (T^2 - T_a^2)}{2} - h_r \frac{WT_a}{2} - g_r \alpha W^\beta \frac{T^2}{2} \quad (A2)$$

and

$$\Pi''_{r2}(T) = -\frac{1}{T^3} \left[2A_r + \frac{(k-h_r)W^{2-\beta}}{\alpha} \right] < 0$$

Hence, $\Pi'_{r2}(T)=0$ has a unique solution T_2^* on $(0, \infty)$.

(c) From (13), we have $\Pi'_{r3}(T)=f_3(T)/T^2$

$$f_3(T) = A_r - \frac{K\alpha W^\beta T^2}{2} + \frac{(k-h_r)WT_a}{2} + \frac{1}{2}(s_r - g_r)\alpha W^\beta M^2 \quad (A3)$$

and

$$\Pi''_{r3}(T) = -\frac{1}{T^3} \left[2A_r + (k-h_r)WT_a + (s_r - g_r)\alpha W^\beta M^2 \right] < 0$$

Hence, $\Pi'_{r3}(T)=0$ has a unique solution T_3^* on $(0, \infty)$.

(d) In a similar argument in (a), Eq.(14) yields,

$\Pi'_{r4}(T)=f_4(T)/T^2$, where

$$f_4(T) = A_r - \frac{K\alpha W^\beta T^2}{2} + \frac{(k-h_r)WT_a}{2} + \frac{1}{2}(s_r - g_r)\alpha W^\beta M^2 \quad (A4)$$

and

$$\Pi''_{r4}(T) = -\frac{1}{T^3} \left[2A_r + (k-h_r)WT_a + (s_r - g_r)\alpha W^\beta M^2 \right] < 0$$

Hence, $\Pi'_{r4}(T)=0$ has a unique solution T_4^* on $(0, \infty)$

(e) From (16), $\Pi'_{r5}(T)=f_5(T)/T^2$, where

$$f_5(T) = \frac{\beta}{1-\beta}(p-c_r + g_r M)(\alpha T)^{\frac{1}{1-\beta}} + A_r - \frac{1}{2(1-\beta)} H_r \alpha^{\frac{1}{1-\beta}} T^{\frac{2-\beta}{1-\beta}} + \frac{(s_r - g_r) \alpha^{\frac{1}{1-\beta}} T^{\frac{\beta}{1-\beta}} M}{1-\beta} \left(\beta(T-M) + \frac{M}{2} \right)$$

(A5)

and

$$\Pi_{r5}'(T) = -\frac{1}{T^3} \left[\frac{\beta(1-2\beta)}{1-\beta}(p-c_r + g_r M)(\alpha T)^{\frac{1}{1-\beta}} + 2A_r + \frac{\beta H_r (\alpha T)^{\frac{1}{1-\beta}} T}{2(1-\beta)^2} + \frac{(s_r - g_r)(1-2\beta)(\alpha T)^{\frac{1}{1-\beta}} M}{(1-\beta)^2 T} (\beta T + (2-3\beta)M) \right] < 0$$

Hence, $\Pi_{r5}'(T)=0$ has a unique solution T_5^* on $(0, \infty)$

Appendix B. Proof of Theorem 1:

If $\Delta_1 < 0$, $\Pi_{r1}'(T_a) < 0$, since $\Delta_1 > \Delta_2 > \Delta_3 > \Delta_4$, we have $\Pi_{r2}'(T_a) < 0$, $\Pi_{r2}'(M) = \Pi_{r3}'(M) < 0$ and $\Pi_{r3}'(T_a+M) = \Pi_{r4}'(T_a+M) < 0$. Therefore, combining with Lemma 1, it implies that $T_1^* < T_a$, $T_2^* < T_a$, $T_3^* < M$ and $T_4^* < T_a+M$. Thus, if $\Delta_1 < 0$, T_1^* is the maximum point of $\Pi_{r1}(Q)$ over $[0, T_a]$, T_a is the maximum point of $\Pi_{r2}(T)$ over (T_a, M) , M is the maximum point of $\Pi_{r3}(T)$ over $[M, T_a+M]$ and T_a+M is the maximum point of $\Pi_{r3}(T)$ over (T_a+M, ∞) . It means that T_1^* is the maximum point of $\Pi_r(T)$ over $(0, \infty)$. Similarly, T_a^* is the maximum point of $\Pi_r(T)$ over $(0, \infty)$ if $\Delta_1 \geq 0$, $\Delta_2 < 0$, T_2^* is the maximum point of $\Pi_r(T)$ over $(0, \infty)$ if $\Delta_2 \geq 0$, $\Delta_3 < 0$, T_3^* is the maximum point of $\Pi_r(T)$ over $(0, \infty)$ if $\Delta_3 \geq 0$, $\Delta_4 < 0$, T_4^* is the maximum point of $\Pi_r(T)$ over $(0, \infty)$ if $\Delta_4 \geq 0$.

Appendix C. Proof of Theorem 2:

If $\Delta_5 < 0$, $\Pi_{r1}'(M) = \Pi_{r5}'(M) < 0$, since $\Delta_5 > \Delta_6 > \Delta_7 > \Delta_4$, we have $\Pi_{r5}'(T_a) < 0$, $\Pi_{r3}'(T_a) < 0$ and $\Pi_{r3}'(T_a+M) = \Pi_{r4}'(T_a+M) < 0$. Therefore, combining with Lemma 1, it implies that $T_1^* < M$, $T_5^* < M$, $T_3^* < T_a$ and $T_4^* < T_a+M$. Thus, if $\Delta_5 < 0$, T_1^* is the maximum point of $\Pi_{r1}(Q)$ over $[0, T_a]$, M is the maximum point of $\Pi_{r2}(T)$ over (M, T_a) , T_a is the maximum point of $\Pi_{r3}(T)$ over $[T_a, T_a+M]$ and T_a+M is the maximum point of $\Pi_{r3}(T)$ over (T_a+M, ∞) . It means that T_1^* is the maximum point of $\Pi_r(T)$ over $(0, \infty)$. Similarly, T_5^* is the maximum point of $\Pi_r(T)$ over $(0, \infty)$ if $\Delta_1 \geq 0$, $\Delta_2 < 0$, T_a^* is the maximum point of $\Pi_r(T)$ over $(0, \infty)$ if $\Delta_2 \geq 0$, $\Delta_3 < 0$, T_3^* is the maximum point of $\Pi_r(T)$ over $(0, \infty)$ if $\Delta_3 \geq 0$, $\Delta_4 < 0$, T_4^* is the maximum point of $\Pi_r(T)$ over $(0, \infty)$ if $\Delta_4 \geq 0$.

References

[1] Aaronson, D., Bostic, R.W., Huck, P., Townsend, R., 2004. Supplier relationships and small business use of trade credit. *Journal of Urban Economics*, 55, 46-67.
 [2] Haley, C.W., Higgins, R.C., 1973. *Inventory policy and trade financing*. *Management Science*, 20 (4): 464-471.

[3] Goyal, S.K., 1985. Economic order quantity under conditions of permissible delay in payment. *Journal of the Operational Research Society*, 36 (4), 335-338.
 [4] Chung K.J., 1998. A theorem on the determination of economic order quantity under conditions of permissible in payments. *Computers and Operations Research*, 25(1): 49-52.
 [5] Teng J.T., 2002. On the economic order quantity under conditions of permissible delay in payments. *Journal of the Operational Research Society*, 36, 335-338.
 [6] Zhou Y.W., 1997. The Effect of the delayed payment for purchasing cost on the optimal order policy of inventory system. *Systems Engineering Theory and Practice*, 17(4), 116-120.
 [7] Chung K.J. and Liao J.J., 2009. The optimal ordering policy of the EOQ model under trade credit depending on the ordering quantity from the DCF approach. *European Journal of Operational Research*, 196(2): 563-568.
 [8] Teng, J.T., Chang, C.T., Goyal, S.K., 2005. Optimal pricing and ordering policy under permissible delay in payments. *International Journal of Production Economics*, 97, 121-129.
 [9] Sheen G.J., Tsao Y.C., 2007. Channel coordination, trade credit and quantity discounts for freight cost. *Transportation Research Part E: Logistics and Transportation Research*, 43, 112-28.
 [10] Chang, H.J., Hung, C.H., Dye, C.Y., 2001. An inventory model for deteriorating items with linear trend demand under the condition that permissible delay in payments. *Production Planning and Control*, 12, 274-282.
 [11] Chang H.J., Hung C.H., Dye C.Y., 2002. A finite time horizon inventory model with deterioration and time-value of money under the conditions of permissible delay in payments. *International Journal of Systems Science* 33:141-51.
 [12] Liao, H.C., Tsai, C.H., Su, C.T., 2000. An inventory model with deteriorating items under inflation when a delay in payment is permissible. *International Journal of Production Economics*, 63, 207-214
 [13] Min Jie, Zhou Yong-Wu, Zhao Ju, 2010. An inventory model for deteriorating items under stock-dependent demand and two-level trade credit. *Applied Mathematical Modelling*, doi:10.1016/j.apm.2010.02.019
 [14] Hartely, V.R., 1976. *Operations Research—A Managerial Emphasis*. Good Year, Santa Monica, CA, pp. 315-317 (Chapter 12).
 [15] Bhunia, A.K., Maiti, M., 1998. A two-warehouse inventory model for deteriorating items with a linear trend in demand and shortages. *Journal of Operational Research Society*, 49, 287-292.
 [16] Zhou, Y.W., 1998. An optimal EOQ model for deteriorating items with two warehouses and time-varying demand. *Mathematica Applicata*, 10, 19-23.
 [17] Kar, S., Bhunia, A.K., Maiti, M., 2001. Deterministic inventory model with two levels of storage, a linear trend in demand and a fixed time horizon. *Computers & Operations Research*, 28, 1315-1331.
 [18] Zhou Yong-Wu and Yang Shan-Lin, 2005. A two-warehouse inventory model for items with stock-level-dependent demand rate. *International Journal of Production Economics*, 95: 215-228.
 [19] Huang, Y.F., 2006. An inventory model under two levels of trade credit and limited storage space derived without derivatives. *Applied Mathematical Modeling*, 30, 418-436
 [20] Chang, K.J., Huang, T.S., 2007. The optimal retailer's ordering policies for deteriorating items with limited storage capacity under trade credit financing. *International Journal of Production Economics*, 106, 127-145.